AIAA JOURNAL
Vol. 34, No. 12, December 1996

Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on the Vorticity Jump Across a Shock Wave

Morris P. Isom*
Armonk, New York 10504
and
Iraj M. Kalkhoran†
Polytechnic University, Brooklyn, New York 11201

RECENT experimental and numerical studies involving the interaction of three-dimensional curved vortices with shock fronts, in a class of problems known as shock wave/vortex interactions, have renewed interest in the evaluation of the vorticity jump across a gasdynamic discontinuity. The original study for inviscid flow, with no other limitations on the state of the flow upstream of the discontinuity, is due to Hayes.[1] In Ref. 1, Hayes used an elegant analytical approach to obtain relations for the vorticity jump across a gasdynamic discontinuity in both steady and unsteady flows. In a recent book, Emanuel[2] disputes the validity of equations derived by Hayes, claiming that the analysis presented in Ref. 1 is in error. This requires some clarification, which we address here.

In Ref. 1, Hayes showed that the normal and tangential components of the vorticity jump across a general three-dimensional shock in a steady flow are

$$\delta\zeta_n = 0 \tag{1}$$

$$\delta\zeta_t = \boldsymbol{n} \times \left[\nabla_t(\rho q_n)\delta(\rho^{-1}) - (\rho q_n)^{-1}\boldsymbol{q}_t \cdot \nabla_t \boldsymbol{q}_t \delta(\rho)\right] \tag{2}$$

where $\zeta = \zeta_t + \boldsymbol{n}\zeta_n$ is the vorticity vector, $\boldsymbol{q}$ the velocity vector, ρ the fluid density, $\boldsymbol{n}$ the unit vector normal to the shock surface, ∇_t the surface gradient operator, and subscripts t and n the components tangential and normal to the shock surface, respectively. Since Eq. (1) indicates that the normal component of the vorticity vector is conserved, the subscript t on ζ in Eq. (2) may be dropped.

In Ref. 2, it is stated that Eq. (1) in Ref. 1, i.e., the vector identity $\nabla \times \boldsymbol{n} = 0$, is in error (see Ref. 2, p. 124). The fact that the vector identity $\nabla \times \boldsymbol{n} = 0$ in general does not hold is irrelevant for the analysis in Ref. 1. In contrast to the discussion in Ref. 2, in the local parallel surface coordinate system (with the shock surface used as a director surface) used by Hayes, the vector identity $\nabla \times \boldsymbol{n} = 0$ is indeed correct. To see this, let ∂n be a differential distance normal to the shock. The gradient operator ∇ is split into vector components tangent and normal to the shock surface by

$$\nabla = \nabla_t + \boldsymbol{n}\frac{\partial}{\partial n}$$

from which the curl of $\boldsymbol{n}$ is

$$\nabla \times \boldsymbol{n} = \nabla_t \times \boldsymbol{n} + \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial n}$$

An elementary theorem in differential geometry is that $\nabla_t \times \boldsymbol{n} = 0$ on any surface for which $\boldsymbol{n}$ is the unit normal (see, for example, Ref. 3, p. 223). Moreover, the trajectories orthogonal to a family of parallel surfaces are straight lines along which $\partial\boldsymbol{n}/\partial n = 0$; hence $\nabla \times \boldsymbol{n} = 0$. Thus, in contrast to claims of Ref. 2, Eq. (1) in Ref. 1 and the preceding equations (1) and (2) are correct as they stand. Note that although Hayes used a parallel surface coordinate system in the neighborhood of the shock to arrive at his results, the derived equations are independent of the coordinate system.

Reference 2 goes on by presenting an equation for the vorticity vector behind the shock, ζ_2 [see Ref. 2, p. 123, Eq. (6.81)] and states that it is different from the equation derived by Hayes and attributes the difference to the aforementioned error in the analysis by Hayes. This claim is also incorrect. First, Ref. 1 does not present a general expression for ζ_2 that is comparable to Eq. (6.81) of Ref. 2. Second, the objective of Ref. 1 is to obtain an expression for the vorticity jump (i.e., $\delta\zeta$) across a general three-dimensional shock front, and the vorticity vectors ζ_1 and ζ_2 are not separately of interest. The most critical and serious claim in Ref. 2 is that Refs. 1 and 4 state that ζ_2 is tangential to the shock's surface (see Ref. 2, p. 123). This is simply not the case. In Refs. 1 and 4 it is not stated anywhere that the vorticity vector behind the shock is generally tangential to the shock's surface. If the flow upstream of the shock is irrotational or the vorticity vector upstream of the shock is tangential to the shock, ζ_2 will indeed be tangential to the shock surface. Except for these cases, ζ_2 will not be tangential to the shock.

An important consequence of deriving the vorticity jump relations in a shock-based parallel surface coordinate system is the explicit dependence of $\delta\zeta$ on the shock curvature tensor $\bar{\bar{\mathfrak{R}}} = -\nabla_t\boldsymbol{n}$ when the upstream flow is uniform. This symmetric 2×2 tensor contains essential properties of the shock geometry. Let $\mathcal{M}$ be the mean curvature and $\mathcal{K}$ the Gaussian curvature of any point on the shock. The trace of $\bar{\bar{\mathfrak{R}}}$ is $2\mathcal{M}$ and its determinant is $\mathcal{K}$. Both $\mathcal{M}$ and $\mathcal{K}$ are scalar invariants, with numerical values independent of the choice of parametric curves used to evaluate $\bar{\bar{\mathfrak{R}}}$ on the shock. It is clear that the shock-based parallel surface frame is the natural coordinate system for the derivation of shock derivatives and vorticity jump relations. A different reference frame would only obscure the relation between $\delta\zeta$ and shock geometry.

Finally, the truly important contribution (and the really deep result) of Ref. 1 is that the vorticity jump across a general three-dimensional gasdynamic discontinuity is a purely dynamical relation obtainable without recourse to the first law of thermodynamics (or Crocco's vorticity theorem). Thus, the relations obtained by Hayes generalize those obtained by Truesdell and Lighthill (see cited references in Ref. 1) for the vorticity behind a steady curved shock in a uniform flow.

References

[1]Hayes, W. D., "The Vorticity Jump Across a Gasdynamic Discontinuity," *Journal of Fluid Mechanics*, Vol. 2, 1957, pp. 595–600.

[2]Emanuel, G., "Normal Derivatives," *Analytical Fluid Dynamics*, CRC Press, Boca Raton, FL, 1994, pp. 121–124.

[3]Aris, R., "The Geometry of Surfaces in Space," *Vectors, Tensors, and Basic Equations of Fluid Mechanics*, Prentice–Hall, Engelwood Cliffs, NJ, 1962, pp. 193–225.

[4]Serrin, J., "Mathematical Principles of Classical Fluid Mechanics," *Handbuch der Physik*, edited by S. Flugge and C. Truesdell, Vol. 8/1, Springer–Verlag, Berlin, 1959, pp. 125–263.

Received April 24, 1996; revision received Aug. 18, 1996; accepted for publication Aug. 18, 1996; also published in *AIAA Journal on Disc*, Volume 2, Number 1.

*Consultant. Member AIAA.

†Associate Professor, Department of Mechanical, Aerospace, and Manufacturing Engineering. Member AIAA.